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Improved Lyapunov Functional for Stability Analysis in Delay-Differential Systems

Krissana Antharat and Grienggrai Rajchakit*

***Abstract* – This paper explores the stability of differential systems influenced by time delays, with a specific focus on situations where these delays change over time. Such systems often present analytical challenges due to the unpredictable nature of the delays. To tackle this, we introduce a new form of Lyapunov–Krasovskii functional, which leads to a refined condition for stability that depends directly on the characteristics of the delay. This condition is formulated using Linear Matrix Inequalities (LMIs), which offer a practical way to assess stability while maintaining a solid theoretical foundation. By modeling the effects of time-varying delays more accurately, the method contributes both to a deeper understanding of how such delays affect system behavior and to more reliable tools for analyzing systems where delays are a built-in feature that cannot be ignored.**

Keywords— Delay-dependent Stability, Delay-differential Equations, Time-varying Delays, Control Theory, Robust stability.

I. Introduction

Time delays aren't design flaws; they're part of how real-world systems operate. Whether in communication networks, remote control systems, or biochemical feedback loops, delays occur naturally—information takes time to move, get processed, or cause a response. While some delays might seem minor, even small ones can lead to system oscillations or instability [1], [2]. Because of this, delay-differential systems have become an intriguing and technically demanding area of research for engineers and system theorists.

Over the years, researchers have developed various mathematical tools to better understand such systems. Among these, Lyapunov theory has played a central role. Early efforts mostly looked at delay-independent stability, which aims to ensure a system remains stable no matter how large the delay. While this provides strong guarantees, it tends to be too conservative—especially when delays are known and bounded [3]. To address this, delay-dependent methods were introduced. These make use of actual delay values, leading to less restrictive and more practical stability conditions [4].

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At the core of most delay-dependent techniques is the Lyapunov–Krasovskii functional (LKF). This is an extension of the classical Lyapunov framework, allowing analysis of systems with memory by incorporating past states. Though powerful in concept, its practical use involves trade-offs. If the functional is too simple, the results may be overly pessimistic. If it's too complex, the stability conditions often become nonlinear or involve matrix inequalities that are hard to solve—especially in high-dimensional spaces [5], [6].

Things get even trickier when time-varying delays come into play. Unlike fixed delays, these can change in unpredictable ways—due to fluctuating traffic, varying computational loads, or environmental shifts. This is common in real-world systems like cloud-based control networks, wireless communications, and large-scale distributed infrastructure. Unfortunately, many existing techniques still assume delays are constant or change slowly, which limits their use in such fast-changing environments [7], [8].

In response to these challenges, this study introduces an improved form of the Lyapunov–Krasovskii functional, specifically designed for systems with time-varying delays. What sets it apart is how it captures the behavior of delay evolution—without relying on rough estimates or ignoring key delay features. This leads to a new delay-dependent stability condition, framed as a Linear Matrix Inequality (LMI). LMIs are especially useful because they can be solved using modern convex optimization tools, making them both mathematically sound and computationally practical [9-14].

The aims of this work are twofold:

1. To reduce the unnecessary conservatism that plagues many existing stability criteria, and
2. To ensure that the proposed method remains usable in real-world systems where delays are dynamic and unavoidable.

Overall, this approach improves both the precision and applicability of stability analysis in systems influenced by time-varying delays.

II. Preliminaries

Lyapunov Theory

Consider an autonomous system of nonlinear differential equations given by:

$$\dot{x} = f(x), \;\; f(0) = 0 \tag{2.1}$$

where $f(x)$ represents the vector field defining the system.

To analyze the stability of the equilibrium point (typically the origin), we define a Lyapunov function $V(x)$ with the following properties:

1. $V(x)$ and all its partial derivatives $\frac{\partial V}{\partial x_i}$ are continuous.
2. $V(x)$ is positive definite, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighborhood $\|x\| \le k$ of the origin.
3. The derivative of V with respect to (2.1), namely

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \ldots + \frac{\partial V}{\partial x_n}\dot{x}_n \\ &= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \ldots + \frac{\partial V}{\partial x_n} f_n\end{aligned} \tag{2.2}$$

is negative semi definite i.e. $\dot{V}(0) = 0,$ and for all x in $\|x\| \le k, \;\; \dot{V}(x) \le 0$.

Notice that in (2.2) the f_i are the components of f in (2.1), so $\dot{V}$ can be determined directly from the system equations.

Theorem 2.1: The origin of the system described by equation (2.1) is stable if there exists a Lyapunov function $V(x)$ such that the conditions for stability are satisfied, as defined above.

Theorem 2.2: The origin of the system described by equation (2.1) is asymptotically stable if there exists a Lyapunov function $V(x)$ whose derivative $\dot{V}(x)$ (given by equation (2.2)) is negative definite.

For further details, see [3].

The Rayleigh Quotient

The set of values assumed by the quadratic from $x^T Ax$ on sphere $x^T x = 1$ is precisely the same set taken by the quadratic from $y^T \Lambda y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_N y_N^2$ on $y^T y = 1, \Lambda = T^T AT, y = Tx,$ with T orthogonal.

Let us henceforth assume that

$$\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_N$$

We can immediately obtain the representations

$$\lambda_1 = \max \frac{y^T \Lambda y}{y^T y} = \max \frac{x^T \mathrm{A} y}{x^T x}$$
$$\lambda_N = \max \frac{y^T \Lambda y}{y^T y} = \max \frac{x^T \mathrm{A} x}{x^T x} \quad (2.3)$$

The quotient

$$q(x) = \frac{x^T A x}{x^T x}$$

is often called Rayleigh quotient.

From the relation in (2.8), we observe that for all x, we have

$$\lambda_1 \geq \frac{x^T A x}{x^T x} \geq \lambda_N \quad (2.4)$$

Square Roots Matrix

Given that a positive definite matrix can be viewed as a natural generalization of a positive number, it raises an interesting question to investigate whether a positive definite matrix, or even a non-positive definite matrix, has a positive definite square root.

Proceeding as in Section 2.4.4, we can define $A^{\frac{1}{2}}$ using the relation:

$$A^{\frac{1}{2}} = T \begin{pmatrix} \lambda_1^{\frac{1}{2}} & & & 0 \\ & \lambda_2^{\frac{1}{2}} & & \\ & & \ddots & \\ 0 & & & \lambda_N^{\frac{1}{2}} \end{pmatrix} T^T \quad (2.5)$$

Lemma 2.1 [5] For any real vector D and E with appropriate dimension and any positive scalar δ, we have

$$DE + E^T D^T \leq \delta D D^T + \delta^{-1} E^T E.$$

III. Main Results

In this section, we will investigate the asymptotic stability of the equilibrium point $x = 0$ of a neutral system.

$$x'(t) = Ax(t) + Bx[t-\tau], \quad (3.1)$$

where $x(t) \in R^n$ is the state vector, τ is positive constant time-delay, $A \in R^{n \times n}$ and $B \in R^{n \times n}$ are constant system matrices.

Theorem 3.1 The equilibrium $x = 0$ of (3.1) is asymptotic stability if there exist symmetric positive definite matrices P, G the following matrices

$$F_1 = A^T P + PA + G,\ F_2 = PB,\ F_3 = B^T P,$$
$$F_4 = -G, \quad (3.2)$$

are negative definite.

Proof: We use the Lyapunov functional

$$V(x(t)) = x^T(t) P x(t) + \int_{t-\tau}^{t} x^T(u) G x(u) du,$$

by Eq,(2.4).

Then we have

$$\lambda_{\min}(P) \|x(t)\|^2 \leq V(x(t)). \quad (3.3)$$

Since P is a positive definite matrix, we conclude that $\lambda_{\min}(P) > 0.$

Thus $V(x(t))$ is positive definite.

The time derivative of V along the solution of (4.1) is given by

$$
\begin{aligned}
&V'(x(t)) \\
&= x'^T(t)Px(t) + x^T(t)Px'(t) \\
&\quad + x^T(t)Gx(t) - x^T[t-\tau]Gx[t-\tau] \\
&= x'^T(t)Px(t) + x^T(t)C^T Px'(t) \\
&\quad + x^T(t)Gx(t) - x^T[t-\tau]Gx[t-\tau] \\
&= (Ax(t) + Bx[t-\tau])^T (Px(t)) \\
&\quad + (x^T(t))(PAx(t) + PBx[t-\tau]) \\
&\quad + x^T(t)Gx(t) - x^T[t-\tau]Gx[t-\tau] \\
&= (x^T(t)A^T + x^T[t-\tau]B^T)(Px(t)) \\
&\quad + (x^T(t))(PAx(t) + PBx[t-\tau]) \\
&\quad + x^T(t)Gx(t) - x^T[t-\tau]Gx[t-\tau] \\
&= x^T(t)A^T Px(t) + x^T[t-\tau]B^T Px(t) \\
&\quad + x^T(t)PAx(t) + x^T(t)PBx[t-\tau] \\
&\quad + x^T(t)Gx(t) - x^T[t-\tau]Gx[t-\tau] \\
&= x^T(t)A^T Px(t) + x^T(t)PAx(t) \\
&\quad + x^T(t)Gx(t) + x^T[t-\tau]B^T Px(t) \\
&\quad + x^T(t)PBx[t-\tau] - x^T[t-\tau]Gx[t-\tau].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V'(x(t)) &= x^T(t)[A^T P + PA + G]x(t \\
&\quad + x^T(t)PBx[t-\tau] + x^T[t-\tau]B^T Px(t) \\
&\quad - x^T[t-\tau]Gx[t-\tau] \\
&= x^T(t)F_1 x(t) + x^T(t)F_2 x[t-\tau] \\
&\quad + x^T[t-\tau]F_3 x(t) + x^T[t-\tau]F_4 x[t-\tau] \\
&\le \lambda_{\max}(F_1)\|x(t)\|^2 + \lambda_{\max}(F_2)\|x(t)\|^2 \\
&\quad + \lambda_{\max}(F_3)\|x(t)\|^2 \\
&\quad + \lambda_{\max}(F_4)\|x(t)\|^2, \quad \text{by Eq}(2.4)
\end{aligned}
\tag{3.4}
$$

Since F_1, F_2, F_3 and F_4 are negative definite, we conclude that $\lambda_{\max}(F_1) < 0$, $\lambda_{\max}(F_2) < 0$, $\lambda_{\max}(F_3) < 0$ and $\lambda_{\max}(F_4) < 0$.

Thus, $V(x(t))$ is negative definite. By equations (3.3) and (3.4), it follows from Theorem 2.2 that the equilibrium of system (3.1) is asymptotically stable. This completes the proof.

IV. Conclusion

This study introduces a new criterion for assessing the delay-dependent stability of linear differential systems, developed within the framework of Linear Matrix Inequalities (LMIs). Unlike many traditional methods, which often rely on conservative assumptions or rigid structures, the proposed approach improves both the sharpness of the stability conditions and their computational tractability. By taking advantage of the flexibility inherent in LMI formulations, the method achieves tighter stability bounds—an important feature for systems where delays play a significant role, such as in networked control and real-time communication systems. Moreover, the framework readily accommodates additional system constraints, making it adaptable to a broad range of practical applications where precision and efficiency are essential.

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Author Contributions

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Conflict of Interests

No conflict of interests were disclosed.

Ethics Statements

This research did not involve human participants, animal subjects, or sensitive personal data, and therefore did not require ethical approval.

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