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A Study on Formulation of Strain-Based Triangular Elements

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Abstract **- This paper presents formulation of triangular elements based on strain approach. Various formulations of the strain-based elements can be seen in the literature. However, most of them do not clearly describe selection of the suitable strain fields for the elements. Therefore, in this work, some guidance for the formulation of strain-based elements are provided based on simulation results that are obtained by using several polynomial functions from Pascal's triangle. It is seen that higher order terms in the strain functions do not always guarantee better performance for the case considered in this work.**

Keywords—strain-based elements, triangular elements, polynomial functions, Pascal's triangle, compatibility, convergence

I. INTRODUCTION

Conventional finite elements are formulated by using displacement-based functions. Displacements within the element are represented by functions (shape functions) which should satisfy certain criteria such as continuity, completeness and possess Kronecker delta property. In [1-3], the authors introduced a new approach for finite element formulation (curved/shell elements) by using functions that define the variation of strain components within the finite element, instead of the displacements.

The strain-based elements are found to be more advantageous compared to the displacement-based elements in terms of convergence and accuracy. Other than that, locking phenomena and errors due to mapping are avoided in strain-based elements [4]. Since then, new strain-based elements have started to emerge, particularly in the form of plane elements for plane elasticity problems [4, 5]. Application of the strain-based elements can also be seen in plate bending [6, 7], linear and dynamic analysis [8, 9], fracture mechanics [10] and so on.

Various formulations of the strain-based elements can be seen in the literature [11]. Functions that are used to define variation of the strain components within the strain-based element are derived based on Pascal's triangle, polynomial Taylor expansion [10], Airy stress functions [12], theories from solid mechanics and by other means. The strain functions should satisfy compatibility equation:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \tag{1}$$

Recent works have shown that incorporating equilibrium conditions into the formulation will yield better performances for the elements [13].

Method for defining the strain fields by using suitable functions are explained in [10] and [12], which are done by using polynomial Taylor expansion and Airy stress functions, respectively. However, most of the other literatures (using polynomial terms from the Pascal's triangle) do not clearly describe selection of the suitable strain fields for the elements [11]. It is also known that accuracy and convergence of the solution can be improved by including the rotational degree of freedom. Nevertheless, very little work has been reported in formulation of low order triangular

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strain-based elements (without rotational degree of freedom) by using higher order terms. Therefore, in this work, an attempt is made to formulate strain-based triangular elements (with 3 nodes and 6 degrees of freedom) by using various polynomial functions from Pascal's triangle.

The triangular elements that are formulated in this work are then used for the analysis of a deep cantilever beam with point load at free end (the other end is fully constrained). The deep cantilever beam is often used as benchmark problem to validate new element formulations. Convergence and accuracy of the elements are observed and compared with the conventional (displacement-based) constant strain triangular (CST) element. This paper is arranged as follows. Methodology and simulation setup for formulation of strain-based elements are provided in section II. Development of strain functions are presented in section III and finally section IV concludes the paper.

II. Methodology

A. Strain Functions

The strain displacement relationships for in plane elasticity are given by:

$$\varepsilon_x = \frac{\partial U}{\partial x} \tag{2a}$$

$$\varepsilon_y = \frac{\partial V}{\partial y} \tag{2b}$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \tag{2c}$$

The strain components in Eq. (2) are then represented by polynomial functions that satisfy Eq. (1). These functions are represented in terms of coefficients a_i (i = 1, 2, … , n). Total number of the coefficient a is equal to the total number of degrees of freedom for the entire triangular element, n. The displacement fields consist of two parts, which are the rigid body mode and the straining mode. The first 3 coefficients (a_1, a_2 *and* a_3) are used to represent the rigid body modes while the rest are used to describe the straining mode.

The rigid body mode is represented by zero strain components within the element [14]:

$$\frac{\partial U}{\partial x} = 0 \tag{3a}$$

$$\frac{\partial V}{\partial y} = 0 \tag{3b}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 0 \tag{3c}$$

Performing the integrations in the Eqs. (3a) and (3b) yield the following:

$$U = a_1 + f(y) \tag{4a}$$

$$V = a_2 + f(x) \tag{4b}$$

Substituting Eqs. (4a) and (4b) into Eq. (3c) yields the following:

$$f_y' + f_x' = 0 \tag{5}$$

Letting $f_y' = -a_3$ and $f_x' = a_3$ yields the following:

$$f_y = -a_3 y \tag{6a}$$

$$f_x = a_3 x \tag{6b}$$

Substituting Eqs. (6a) and (6b) into Eqs. (4a) and (4b) result in the displacement fields for the rigid body mode:

$$U = a_1 - a_3 y \tag{7a}$$

$$V = a_2 + a_3 x \tag{7b}$$

Remaining 3 coefficients (a_4, a_5 *and* a_6) are used to define the straining mode of the displacement field. Several polynomial functions (that satisfy Eq. (1)) are developed to represent the strain components (straining mode) for a triangular element with 3 nodes and 2 degree of freedoms per node (n = 6) as shown in Table I. The functions for the straining mode are developed by ensuring that they will yield displacement functions with complete and balanced terms from the Pascal's triangle.

Table I. Strain Functions for the Formulation of Strain-based Triangular Elements.

Test	Strain functions (for straining mode)	Complete displacement functions
T1	$\varepsilon_x = a_4$ $\varepsilon_y = a_5$ $\gamma_{xy} = a_4 + a_5 + a_6$	$U = a_1 - ya_3 + \left(x + \frac{y}{2}\right)a_4 + \left(\frac{y}{2}\right)a_5 + \left(\frac{y}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{x}{2}\right)a_4 + \left(\frac{x}{2} + y\right)a_5 + \left(\frac{x}{2}\right)a_6$
T2	$\varepsilon_x = a_4$ $\varepsilon_y = a_5$ $\gamma_{xy} = a_6$	$U = a_1 - ya_3 + (x)a_4 + \left(\frac{y}{2}\right)a_6$ $V = a_2 + xa_3 + (y)a_5 + \left(\frac{x}{2}\right)a_6$
T3	$\varepsilon_x = a_4 + a_5 + a_6$ $\varepsilon_y = a_4 + a_5 + a_6$ $\gamma_{xy} = a_4 + a_5 + a_6$	$U = a_1 - ya_3 + \left(x + \frac{y}{2}\right)a_4 + \left(x + \frac{y}{2}\right)a_5 + \left(x + \frac{y}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{x}{2} + y\right)a_4 + \left(\frac{x}{2} + y\right)a_5 + \left(\frac{x}{2} + y\right)a_6$
T4	$\varepsilon_x = a_4 + a_5 x + a_6 y$ $\varepsilon_y = a_4 + a_5 x + a_6 y$ $\gamma_{xy} = a_4 + a_5 x + a_6 y$	$U = a_1 - ya_3 + \left(x + \frac{y}{2}\right)a_4 + \left(\frac{x^2}{2} - \frac{y^2}{2}\right)a_5 + \left(xy + \frac{y^2}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{x}{2} + y\right)a_4 + \left(\frac{x^2}{2} + xy\right)a_5 + \left(\frac{y^2}{2} - \frac{x^2}{2}\right)a_6$
T5	$\varepsilon_x = a_4 x + a_5 x^2$ $\varepsilon_y = a_4 y + a_5 y^2$	$U = a_1 - ya_3 + \left(\frac{x^2}{2} + \frac{y}{2}\right)a_4 + \left(\frac{x^3}{3} + \frac{y}{2}\right)a_5 + \left(\frac{y}{2}\right)a_6$

	$\gamma_{xy} = a_4 + a_5 + a_6$	$V = a_2 + xa_3 + \left(\frac{x}{2} + \frac{y^2}{2}\right)a_4 + \left(\frac{x}{2} + \frac{y^3}{3}\right)a_5 + \left(\frac{x}{2}\right)a_6$
T6	$\varepsilon_x = a_4x + a_5x^2$ $\varepsilon_y = a_4y + a_5y^2$ $\gamma_{xy} = a_6$	$U = a_1 - ya_3 + (x)a_4 + \left(\frac{x^3}{3}\right)a_5 + \left(\frac{y}{2}\right)a_6$ $V = a_2 + xa_3 + (y)a_4 + \left(\frac{y^3}{3}\right)a_5 + \left(\frac{x}{2}\right)a_6$
T7	$\varepsilon_x = a_4 + a_5x + a_6y$ $\varepsilon_y = a_4 + a_5x + a_6y$ $\gamma_{xy} = a_6$	$U = a_1 - ya_3 + (x)a_4 + \left(\frac{x^2}{2} - \frac{y^2}{2}\right)a_5 + \left(xy + \frac{y}{2}\right)a_6$ $V = a_2 + xa_3 + (y)a_4 + (xy)a_5 + \left(\frac{x}{2} - \frac{x^2}{2} + \frac{y^2}{2}\right)a_6$
T8	$\varepsilon_x = a_4 + a_5x$ $\varepsilon_y = a_4y + a_5$ $\gamma_{xy} = a_6$	$U = a_1 - ya_3 + (x)a_4 + \left(\frac{x^2}{2}\right)a_5 + \left(\frac{y}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{y^2}{2}\right)a_4 + (y)a_5 + \left(\frac{x}{2}\right)a_6$
T9	$\varepsilon_x = a_4x^2 + a_5x$ $\varepsilon_y = a_4y + a_5y^2$ $\gamma_{xy} = a_6$	$U = a_1 - ya_3 + \left(\frac{x^3}{3}\right)a_4 + \left(\frac{x^2}{2}\right)a_5 + \left(\frac{y}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{y^2}{2}\right)a_4 + \left(\frac{y^3}{3}\right)a_5 + \left(\frac{x}{2}\right)a_6$
T10	$\varepsilon_x = a_4x^2 + a_5x$ $\varepsilon_y = a_4y + a_5y^2$ $\gamma_{xy} = a_4 + a_5x + a_6y$	$U = a_1 - ya_3 + \left(\frac{x^3}{3} + \frac{y}{2}\right)a_4 + \left(\frac{x^2}{2}\right)a_5 + \left(\frac{y^2}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{x}{2} + \frac{y^2}{2}\right)a_4 + \left(\frac{x^2}{2} + \frac{y^3}{3}\right)a_5$
T11	$\varepsilon_x = a_4x^2 + a_5x$ $\varepsilon_y = a_4y + a_5y^2$ $\gamma_{xy} = a_5x + a_6y$	$U = a_1 - ya_3 + \left(\frac{x^3}{3}\right)a_4 + \left(\frac{x^2}{2}\right)a_5 + \left(\frac{y^2}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{y^2}{2}\right)a_4 + \left(\frac{x^2}{2} - \frac{y^3}{3}\right)a_5$
T12	$\varepsilon_x = a_4x^2 + a_5(1 + x)$ $\varepsilon_y = a_4(1 + y) + a_5y^2$ $\gamma_{xy} = a_4 + a_5x + a_6y$	$U = a_1 - ya_3 + \left(\frac{x^3}{3} + \frac{y}{2}\right)a_4 + \left(x + \frac{x^2}{2}\right)a_5 + \left(\frac{y^2}{2}\right)a_6$ $V = a_2 + xa_3 + \left(\frac{x}{2} + y + \frac{y^2}{2}\right)a_4 + \left(\frac{x^2}{2} + \frac{y^3}{3}\right)a_5$

Displacements functions (for straining mode) are obtained by substituting the strain functions (second column of Table I) into Eqs. (2a) and (2b) and perform the integration. The integration will generate unknown functions *f(x)* for *V* and *f(y)* for *U*. Derivatives of these unknown functions, that are *f'(x)* and *f'(y)* are obtained by substituting the *U* and *V* equations (with *f(x)* and *f(y)* after the integration) into Eq. (2c). Resultant equation will be the expression for shear strain γ which is given in terms of *f'(x)* and *f'(y)*. Another expression for shear strain γ is obtained by substituting the strain functions in Table I into Eq. (2c). These two different expressions for the same shear strain γ are then compared. The functions *f(x)* and *f(y)* are finally obtained by integrating the *x* and *y* terms that are collected through the comparison of the two expressions. The complete expressions (containing both rigid body and straining modes) for the displacement fields *U* and *V* are provided in Table I.

The stiffness matrix for a strain-based triangular element is obtained by using the formula:

$$K_e = Q^T\left[\iint_\Omega (B^T DB)d\Omega\right]Q \tag{8}$$

where *Q* is the transformation matrix:

$$Q = \begin{bmatrix} U(x_1, y_1) \\ V(x_1, y_1) \\ U(x_2, y_2) \\ V(x_2, y_2) \\ U(x_3, y_3) \\ V(x_3, y_3) \end{bmatrix} \tag{9}$$

B is the 3 by 6 strain matrix which consists of coefficients for a_i in the complete strain functions (containing both the rigid body and straining modes). The complete strain functions are obtained by substituting the complete displacement functions (third column of Table I) into Eqs. (2a)-(2c). For example, the *B* matrix for T_1 of Table I is given as:

$$B_{T_1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{10}$$

D is the constitutive matrix:

$$D = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & 1-v \end{bmatrix} \text{ for plane stress} \tag{11}$$

and Ω is the element domain.

The integration in Eq. (8) can be performed easily to obtain high accuracy, since the integrand consists of simple polynomial terms which are derived from the Pascal's triangle. In case of fully numerical integration method, the random triangular domain is first mapped to a reference triangle (fixed geometry) and then the integration is performed by using the Gaussian integration points and weights within the reference triangle [15]. Exact integration can be performed by using the techniques presented in [16, 17]. In this work, the integration is performed by using line integral [18], which is a semi-analytical technique [17]. It is considered as a semi analytical technique, since the first integration is done analytically (with respect to *x*) and through divergence theorem, yields the following expression [18]:

$$(y_2 - y_1)\int_0^1 f[x_1 + \tau(x_2 - x_1), y_1 + \tau(y_2 - y_1)]d\tau \tag{12}$$

The next integration (with respect to *y*) is done numerically by using one dimensional quadrature rule through the formula:

$$I = \int_a^b f(x)w(x)dx = \sum_{i=1}^{n} w_i f(x_i) \tag{13}$$

, where a and b represent integration limits, $f(x)$ represents function to be integrated, $w(x)$ represents weight functions, w_i represents integration weights, xi represents integration points, $i = 1, 2, 3, \ldots, n$, and n represents integration order.

B. Simulation Setup

The strain based triangular elements are tested by analyzing a deep cantilever beam with point load at free end (the other end is fully constrained). Three different meshing are considered, which consists of 20 (mesh 1), 80 (mesh 2) and 320 (mesh 3) elements with 18, 55 and 189 global nodes, respectively. Properties of the beam are: length = 10 m, height = 4 m, thickness = 0.0625 m, modulus of elasticity = 100 MPa and Poisson's ratio = 0.2. The point load, F = 100 kN. Problem geometry and boundary conditions are shown in Fig. 1.

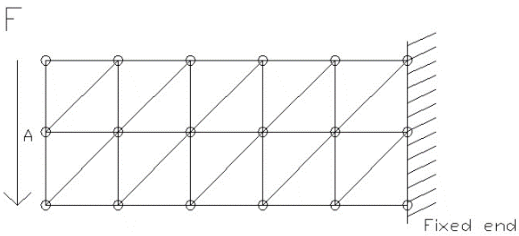

Fig. 1. A deep cantilever beam with point load.

Convergence and accuracy of the elements in calculating the deflection of point A (midpoint of the free end) are obtained by running simulations in Mathematica. Analytical solution for the deflection at point A is 1.105 mm [12]. Simulation results for the elements in Table I are shown and discussed in section III.

III. Development of Strain Functions

Suitable strain functions for the straining mode can be developed through several stages. First attempt was to formulate the strain functions by using constants a_i, without any other polynomial terms. T_1, T_2 and T_3 in Table 1 are such examples. T_1 and T_2 are found to perform exactly similar to CST element (since the strains are assumed to be constant within the element), while T_3 yields singular matrix. It is seen that similar functions for the strain components yield singular matrix. Few more test functions with similar expressions for ε_x and ε_y are formed (T_4 - T_7) and simulated. These functions yield either diverging solutions (T_4) or singular matrices (T_5 - T_7). Simulation results for these functions are shown in Fig. 2.

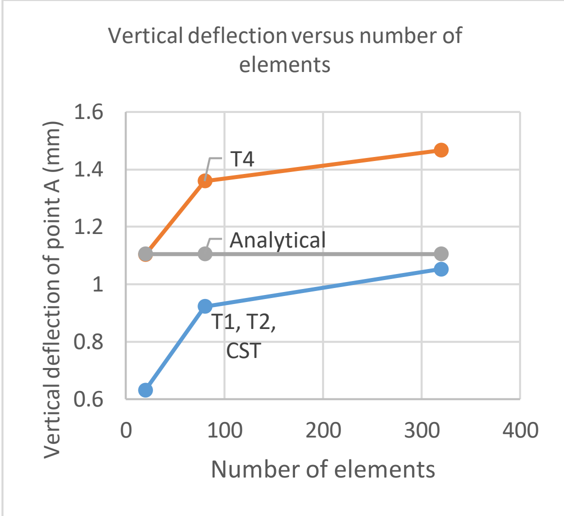

Fig. 2. Simulation results for constant strain functions.

First order terms from Pascal triangle (*constant*, x, y) are then used to formulate T_8. The shear strain γ_{xy} is remained as constant. Similar functions are avoided by distributing the terms among ε_x and ε_y. Performance of T_8 is found to be similar to CST as shown in Fig. 3.

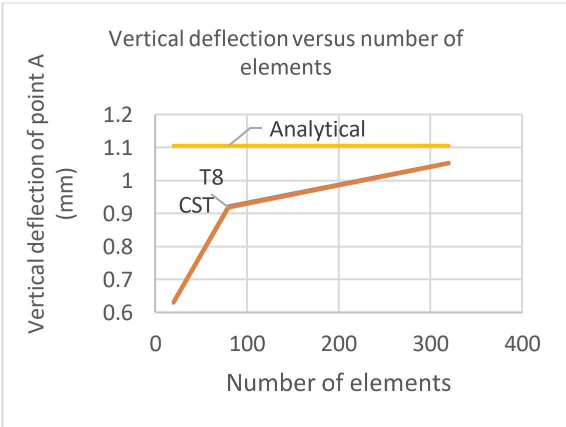

Fig. 3. Simulation results for linear strain functions.

Second order polynomial terms (*Constant*, x, y, x^2 and y^2) are then introduced into the trial functions, T_9. Similar expressions for the strain components are avoided. The shear strain γ_{xy} is remained as constant. T_9 is found to be able to produce converging results. However, the results are less accurate than CST, even with second order terms as shown in Table II.

Table II. Simulation Results for T_9 to T_{12}.

Mesh density	Percentage error for deflection at point A (%)				
	CST	T_9	T_{10}	T_{11}	T_{12}
Mesh 1	42.79	43.01	39.87	60.79	40.07
Mesh 2	16.55	17.23	16.23	70.71	16.57
Mesh 3	4.68	5.02	4.94	15.44	5.20

Another two test functions: T_{10}, and T_{11} are then formulated by considering the previous observations, but first order terms are used to represent the shear strain γ_{xy}, instead of using constant. No significant improvement is seen for T_{10}, even though better accuracy is obtained for lower mesh densities (meshes 1 and 2) as compared to CST as shown in Table II. On the other hand, T_{11} generated diverging solutions for meshes 1 and 2. It is seen that γ_{xy}should contain a constant term in order to achieve good convergence of the solutions.

Finally, test function T_{12} *is* formed by considering the previous observations, but constant terms are included into the strain functions by sharing the coefficient $a_{i.}$ The test function T_{12} produced converging solutions, but did not outperform CST element.

It is seen that higher order terms in the strain functions do not yield better performance compared to CST. One of the reasons is that the polynomial terms are shared among the limited number of coefficients a_i (only 3 in this case). Limited number of coefficients a_i for the straining functions will hinder the ability to form independent complete polynomial terms. Strain-based elements with higher degree of freedom offer better flexibility to form independent and complete polynomial terms, and therefore they are capable to produce better results.

General guidelines for formation of strain-based finite elements by using polynomial terms can then be derived as follows:

1. The strain functions should satisfy Eq. (1). Equilibrium conditions should also be satisfied in order to achieve optimal performances as shown in [13].
2. Similar functions/expressions for the strain components should be avoided in order to prevent singular matrices.
3. The test functions should contain all the balanced terms from Pascal triangle and distributed evenly.

Polynomial strain functions are convenient for strain-based elements, since the integration can be performed with high accuracy. The semi-analytical integration technique is found to be practical and easier, since the technique does not require any partitioning of the element, mapping to a reference triangle or formulation of formulas based on specific forms.

IV. CONCLUSION

Several strain-based triangular elements have been successfully developed and tested by running simulations in Mathematica. It is seen that formulation of the strain based triangular elements with higher order terms in the strain functions do not always guarantee better performance. The techniques/method presented in this work can be used as a guidance in formulating strain-based elements.

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